MIXED CONVECTION FLOW IN HEAT ABSORBING DARCIAN MEDIA WITH VISCOUS DISSIPATION AND THERMAL RADIATION

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Abstract: The present paper deals with the study of a mixed convective flow of Newtonian, incompressible, viscous fluid past a porous vertical plate embedded in heat absorbing Darcian porous media in presence of viscous dissipation and thermal radiation. A system of non-linear, coupled partial differential equations has been developed based on the basic conservation principles of physics such as conservation of mass, conservation of momentum and conservation of energy. Multi – parameter regular perturbation technique has been used to solve the governed system of equations subject to a set of favourable boundary conditions. The exact closed form of solutions for the velocity, temperature as well as skin – friction and Nusselt number have been obtained in terms of some governed physical parameters. Numerical simulations in terms of graphs and table have been made to investigate the effects of pertinent parameters. It is observed that, an increase in Reynolds number decreases the temperature as well as the velocity while the presence of Reynolds number increases the Nusselt number as well as skin- friction. Again the fluid velocity and temperature are found to increase due to increase in Eckert number but a reversed effect has observed in case of Nusselt number and skin – friction.

Keywords: Newtonian fluid, mixed convection, heat transfer, heat absorption, Darcian

Introduction

The phenomena of mixed or combined convection arise when both free and forced convection occur simultaneously in a flow domain. Several researchers investigated the theory of mixed convection due to its applications in many industrial and mechanical devices. The pioneering work in mixed convection flow was made by Ostrich (1954), where he investigated the combined natural and forced convection laminar flow and heat transfer of fluid with and without heat source in channels with linearly varying wall temperature. The case of mixed convection cooling of a heated, continuously stretching surfaces was considered by Chen (1998). Hadhrami *et al.* (2001) justified the study of combined free and forced convection in vertical channels of porous media. Mahmud and Fraser (2003) studied the mixed convection – radiation interaction in a vertical porous channel with entropy generation. Gireesha *et al.* (2013) enunciated the study of mixed convective flow of a dusty fluid over a vertical stretching sheet with non-uniform heat source/sink and radiation. Recently, Dawood *et al*. (2015) reviewed the process of forced, natural and mixed-convection heat transfer and fluid flow in annulus.

On the other hand, radiation heat transfer plays an important role in manufacturing industries for the design of reliable equipment. If the temperature of the surrounding fluid is rather high, radiation effects play an important role. The theory of thermal radiation has been considered by many researchers in various flow situations. Babu *et al*. (2011), analyzed radiation and chemical reaction effects on unsteady MHD convection flow past a vertical moving porous plate embedded in a porous medium with viscous dissipation. Shit and Halder (2011), Ahmed *et al*. (2011), investigated the effects of thermal radiation on MHD viscous fluid flow over a shrinking sheet. The effect of thermal radiation on boundary layer flow and heat transfer of dusty fluid over an unsteady stretching sheet was put forward by Manjunatha and Gireesha (2012), Christophand Johannes (2014), investigated numerically the heat transfer with thermal radiation in an enclosure in case of buoyancy driven flow. Sengupta (2015) investigated the thermal radiation effect with chemical reaction and radiation absorption along with the conditions of variable wall temperature and concentrations. Very recently, Sahoo*et al*. (2016) analyzed thermal radiation heat transfer model and its application for automobile exhaust components. The problem of thermal radiation in unsteady mixed convection flow is attracted attention of many researchers due to some practical applications in engineering, technology and in cooling of nuclear reactors. Researchers like, Elsayed *et al*. (2012), Reddy *et al.* (2013) and Khan *et al*. (2014) developed the theory of thermal radiation on various flow situations. Off late, Devi *et al*. (2016) investigated the effect of radiation on an unsteady MHD mixed convective flow past an accelerated vertical porous plate with suction and chemical reaction.

It is interesting to observe that, if a fluid is highly viscous or fluid shear rate is of high in magnitude, the viscous dissipation is predominant. Viscous dissipation is a mechanical process defined as the heat that produces due to work done by fluid particles on adjacent layers to counter the shearing forces. In case of mixed convection flow, the effect of viscous dissipation is significant and cannot be ignored. The significant contribution in viscous dissipation was made by Gebhart (1962), who had investigated the effect of viscous dissipation in natural convection. Considering the importance of viscous dissipation, Fand and Brucker (1983), Mahajan and Gebhart (1989) investigated viscous dissipation in natural convection and buoyancy induced flows respectively. Barletta (1998), studied laminar mixed convection with viscous dissipation in a vertical channel, Nield *et al*. (2003), considered the viscous dissipation effect in forced convection thermally developing flow through parallel porous channel with walls at uniform temperature. Pentokratoras (2005) investigated the effect of viscous dissipation in natural convection along a heated vertical plate.

Again, the study of viscous dissipation with mixed convection is significant in industrial and technological point of view. Hung and Tao (2009), considered the effects of viscous dissipation on fully developed forced convection flow in porous media. Das (2014) investigated the influence of chemical reaction as well as viscous dissipation on MHD mixed convection flow. Abdollahzadeh and Hyun (2014) undertook the problem of viscous dissipation with thermal radiation and Joule heating for the study on MHD forced convection flow. Pal and Samad (2015) used similarity transformations and obtained numerical solutions for analyzing the combined effects of viscous dissipation and thermal radiation on non-Newtonian fluid along a surface with heat generation and uniform surface heat flux.

Following the importance of the aforesaid phenomena, the purpose of the present paper is to study the flow behavior of a mixed convection flow of a Newtonian, incompressible, viscous fluid through a vertical porous plate immersed in heat absorbing Darcian porous media in presence of thermal radiation and viscous dissipation effects.

Basic equations and assumptions undertaken

Basic equations

The vector forms of equations that describe the flow situation are formulated as:

$$\vec{\nabla}.\vec{q} = 0 \qquad \text{(Continuity Equation)}$$

$$\rho\left(\frac{\partial \vec{q}}{\partial t} + (\vec{q}.\vec{\nabla})\vec{q}\right) = -\vec{\nabla}p - \rho g + \mu\vec{\nabla}^2\vec{q} - \frac{\mu}{K}\vec{q} \qquad \text{(Modified Navier-stokes equation)}$$

$$\rho C_p\left(\frac{\partial T}{\partial t} + (\vec{q}.\vec{\nabla})T\right) = k\vec{\nabla}^2 T - \vec{\nabla}\vec{q}_R + Q^*\delta T \qquad \text{(Energy equation)}$$

$$\rho_\infty - \rho \approx \rho\beta_T(\bar{T} - \bar{T}_\infty) \qquad \text{(Boussinesq approximation)}$$

Where, $\vec{q} = (\bar{u}, \bar{v}, 0)$: the fluid velocity, p : the hydrostatic pressure, ρ: the fluid density, g : the acceleration due to gravity, μ : the dynamic viscosity, k: the thermal conductivity of the medium, $\vec{\nabla}_{(x,y)} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right)$: 2-D vector gradient operator, $\vec{q}_R$: the radiative heat flux, ρ_∞ : the densities of fluid particles in free stream, β_T : coefficient of volumetric thermal expansion, $\bar{T}, \bar{T}_\infty$: Fluid temperature near the plate and in free stream, Q^* : First order heat source parameter, C_p : specific heat at constant pressure.

Basic assumptions

The fundamental assumptions considered for the study are as follows:

a) All the fluid properties except possibly the pressure are independent of variations of x*-scale.
b) All the fluid properties are considered constant except the physical influence of the density term with temperature in the thermal buoyancy force.
c) The empirical relation between the pressure gradient and the flow velocity is assume as linear as such, the Brinkman– Darcy porous term is considered; while Forchheimer's non-linear porous effect is neglected therein.
d) The flow domain is considered as homogenous and isotropic.
e) Due to moderate viscous fluid, the viscous dissipation of energy is considered in the energy equation.
f) The temperature of fluid particles near the plate surface is supposed to be more than their respective components at the free stream region.
g) The radiation heat fluxes are considered prominent along normal to the plate i.e. towards the fluid domain and are thus taken negligible along the radial direction.

Mathematical formulation of the problem

The mixed convective flow of an unsteady, laminar, semi-two dimensional incompressible viscous fluid over an infinite vertical porous flat plate embedded in Darcian porous medium is considered for the study. The positive $\bar{x}$ -coordinate is measured along the plate in the vertically upward direction and the positive $\bar{y}$ - coordinate is taken normal to the plate in the outward direction towards the fluid region.

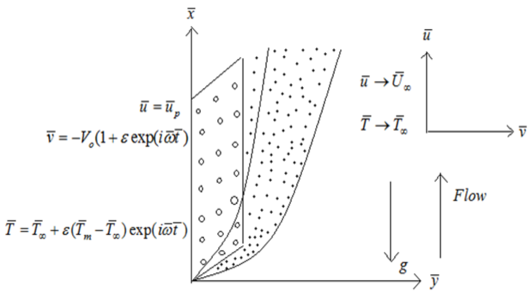

Fig. 1: A schemetic representation of the flows coordinate system

Under the Boussinesq and boundary layer approximations, the basic boundary-layer equations thus governed as:

Continuity equation

$$\frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{1}$$

$\bar{x}$ momentum equation

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{x}} + \upsilon\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \frac{\upsilon}{\bar{K}}\bar{u} + g\beta_T\left(\bar{T} - \bar{T}_\infty\right) \tag{2}$$

Energy equation

$$\frac{\partial \bar{T}}{\partial \bar{t}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \frac{k}{\rho C_p}\frac{\partial^2 \bar{T}}{\partial \bar{y}^2} + \frac{\upsilon}{C_p}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2 - \bar{Q}_s\left(\bar{T} - \bar{T}_\infty\right) - \frac{1}{\rho C_p}\frac{\partial \bar{q}_{r\bar{y}}}{\partial \bar{y}} \tag{3}$$

With use of the relevant boundary condition as:

$\bar{u}=0,\bar{v}=0,\bar{T}=\bar{T}_\infty$ for all $\bar{y}\le 0$ and $\bar{t}\le 0$ (4.1)

$\bar{u}=\bar{u}_p,\bar{v}=-V_o\left(1+\varepsilon\exp\left(i\bar{\omega}\bar{t}\right)\right),\bar{T}=\bar{T}_\infty+\varepsilon\left(\bar{T}_m-\bar{T}_\infty\right)\exp\left(i\bar{\omega}\bar{t}\right)$ at $\bar{y}=0$ when $\bar{t}>0$ (4.2)

$\bar{u}\to\bar{U}_\infty=U_o\left(1+\varepsilon\exp\left(i\bar{\omega}\bar{t}\right)\right),\bar{T}\to\bar{T}_\infty$ for $\bar{y}\to\infty$ when $\bar{t}>0$ (4.3)

Where, $(\bar{u},\bar{v}),\bar{t},\upsilon,\bar{K},\bar{u}_p,\varepsilon,\bar{\omega},V_o,\bar{U}_\infty,U_o$ are respectively the ($\bar{x}$, $\bar{y}$) component velocities, time variable, kinematic co-efficient of viscosity, permeability parameter, slip velocity of the plate, fluctuation parameter (ε << 1), frequency of oscillation, mean plate suction velocity, free stream velocity, mean free stream velocity.

The Bernoullie's pressure equation gives,

$$-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial\bar{y}}=\frac{\upsilon}{\bar{k}}\bar{U}_o+\varepsilon\left(\frac{\upsilon}{\bar{k}}+i\bar{\omega}\right)\bar{U}_o\exp(i\bar{\omega}\bar{t}) \quad (5)$$

Now using (5) in (2) we get,

$$\frac{\partial\bar{u}}{\partial\bar{t}}+\bar{v}\frac{\partial\bar{u}}{\partial\bar{y}}=\upsilon\frac{\partial^2\bar{u}}{\partial\bar{y}^2}+\frac{\upsilon}{\bar{k}}\left(\bar{U}_o-\bar{u}\right)+\varepsilon\left(\frac{\upsilon}{\bar{k}}+i\omega\right)\bar{U}_o\exp(i\bar{\omega}\bar{t})+g\beta_T\left(\bar{T}-\bar{T}_\infty\right) \quad (6)$$

We now use a set of non-dimensional quantities as:

$$x=\frac{\bar{x}}{L},y=\frac{\bar{y}}{L},u=\frac{\bar{u}}{V_o},v=\frac{\bar{v}}{V_o},u_p=\frac{\bar{u}_p}{V_o},t=\frac{\bar{t}V_o}{L},\omega=\frac{\bar{\omega}L}{V_o},\theta=\frac{\bar{T}-\bar{T}_\infty}{\bar{T}_m-\bar{T}_\infty},U_\infty=\frac{\bar{U}_\infty}{V_o},U_o=\frac{\bar{U}_o}{V_o},\ \mathrm{Re}_L=\frac{V_oL}{\upsilon},$$

$$K=\frac{\bar{K}V_o}{\upsilon L},G_r=\frac{g\beta_T\left(\bar{T}_m-\bar{T}_\infty\right)L}{V_o^{\,2}},P_r=\frac{\rho\upsilon c_p}{k},\ \mathrm{Re}_L=\frac{V_oL}{\upsilon},\ Q_s=\frac{\bar{Q}_sL}{V_0},N=\frac{4\sigma_1\bar{T}_\infty^{\,3}}{\rho C_pk_1},\ R=1+\frac{4N}{3}$$

Where, $L,\mathrm{Re}_L,G_r,P_r,Q_S$ and N represents some reference length along the normal scale, local Reynolds number, thermal Grashof number, Prandtl number, heat source/generation parameter, thermal radiation parameter, respectively.

The non - dimensional set of equations on using Rossolandradiative approximation model for optically thick medium is obtained as:

$$\frac{\partial v}{\partial y}=0 \quad (7)$$

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}=\frac{1}{\mathrm{Re}_L}\frac{\partial^2u}{\partial y^2}+\frac{1}{K}\left(U_o-u\right)+\varepsilon\left(\frac{1}{K}+i\omega\right)\exp(i\omega t)U_o+G_r\theta \quad (8)$$

$$\frac{\partial\theta}{\partial t}+v\frac{\partial\theta}{\partial y}=\frac{N}{P_r\,\mathrm{Re}_L}\frac{\partial^2\theta}{\partial y^2}+E_c\left(\frac{\partial u}{\partial y}\right)^2-Q_s\theta \quad (9)$$

With non-dimensional boundary condition as:

$u=0,v=0,\theta=0,$ for $y\le 0$ when $t\le 0$ (10.1)

$u=u_p,v=-\left(1+\varepsilon\exp\left(i\omega t\right)\right),\theta=1+\varepsilon\exp\left(i\omega t\right)$ at $y=0$ when $t>0$ (10.2)

$u\to U_\infty=U_o\left(1+\varepsilon\exp\left(i\omega t\right)\right),\theta\to 0$ for $y\to\infty$ when $t>0$ (10.3)

Method of solution

We use a first set of perturbation solutions as:

$$f\left(y,t\right)=f_o\left(y\right)+\varepsilon f_1\left(y\right)\exp(i\omega t)$$

Where, f stands for θ and u etc.

On using the form, we obtain a set of mean and perturbed equations for temperature as well as velocity fields as:

$$N\frac{d^2\theta_o}{dy^2}-P_r\,\mathrm{Re}_L\,v_o\frac{d\theta_o}{dy}-Q_sP_r\,\mathrm{Re}_L\,\theta_o=E_cP_r\,\mathrm{Re}_L\left(\frac{du_o}{dy}\right)^2 \quad (11)$$

$$N\frac{d^2\theta_1}{dy^2}-P_r\,\mathrm{Re}_L\,v_o\frac{d\theta_1}{dy}-\left(Q_s+i\omega\right)P_r\,\mathrm{Re}_L\,\theta_1=P_r\,\mathrm{Re}_L\,v_1\frac{d\theta_o}{dy}-2E_cP_r\,\mathrm{Re}_L\frac{du_o}{dy}\frac{du_1}{dy} \quad (12)$$

$$\frac{d^2u_o}{dy^2}-v_o\,\mathrm{Re}_L\frac{du_o}{dy}-\frac{\mathrm{Re}_L}{k}u_o=-\frac{\mathrm{Re}_L}{k}U_o-G_r\,\mathrm{Re}_L\,\theta_o \quad (13)$$

$$\frac{d^2u_1}{dy^2}-\mathrm{Re}_L v_o\frac{du_1}{dy}-M_1\mathrm{Re}_L u_1=\mathrm{Re}_L v_1\frac{du_o}{dy}-\mathrm{Re}_L U_o M_1-G_r\mathrm{Re}_L\theta_1 \quad (14)$$

$$\text{where } M_1=\frac{1}{k}+i\omega \quad (\text{say})$$

With the boundary condition as:

$$u_o=u_p, v_o=-1, \theta_o=1, u_1=0, v_1=-1, \theta_1=1 \text{ at } y=0, t>0 \quad (15.1)$$

$$u_o\to U_o, \theta_o\to 0, u_1\to U_o, \theta_o\to 0 \text{ for } y\to\infty,\ t>0 \quad (15.2)$$

We now use a second set of perturbation as:

$$f_o(y)=f_{oo}(y)+E_c f_{01}(y)+o(E_c^{\,2})$$

$$f_1(y)=f_{1o}(y)+E_c f_{11}(y)+o(E_c^{\,2})$$

Using the above trial forms, we obtain a set of unperturbed as well as perturbation parts of equations as:

$$N\frac{d^2\theta_{oo}}{dy^2}-P_r\mathrm{Re}_L v_{oo}\frac{d\theta_{oo}}{dy}-Q_sP_r\mathrm{Re}_L\theta_{oo}=0 \quad (16)$$

$$N\frac{d^2\theta_{01}}{dy^2}-P_r\mathrm{Re}_L v_{oo}\frac{d\theta_{01}}{dy}-Q_sP_r\mathrm{Re}_L\theta_{01}=v_{01}P_r\mathrm{Re}_L\frac{d\theta_{oo}}{dy}-P_r\mathrm{Re}_L\left(\frac{du_{oo}}{dy}\right)^2 \quad (17)$$

$$\frac{d^2u_{oo}}{dy^2}-\mathrm{Re}_L v_{oo}\frac{du_{oo}}{dy}-\frac{\mathrm{Re}_L}{k}u_{oo}=-\frac{\mathrm{Re}_L}{k}U_o-\mathrm{Re}_L G_r\theta_{oo} \quad (18)$$

$$\frac{d^2u_{01}}{dy^2}-\mathrm{Re}_L v_{oo}\frac{du_{01}}{dy}-\frac{\mathrm{Re}_L}{k}u_{01}=\mathrm{Re}_L v_{01}\frac{du_{00}}{dy}-G_r\mathrm{Re}_L\theta_{01} \quad (19)$$

$$N\frac{d^2\theta_{10}}{dy^2}-P_r\mathrm{Re}_L v_{oo}\frac{d\theta_{10}}{dy}-(Q_s+i\omega)P_r\mathrm{Re}_L\theta_{10}=P_r\mathrm{Re}_L v_{10}\frac{d\theta_{oo}}{dy} \quad (20)$$

$$N\frac{d^2\theta_{11}}{dy^2}-\mathrm{P}_r\mathrm{Re}_L v_{oo}\frac{d\theta_{11}}{dy}-(Q_s+i\omega)\mathrm{P}_r\mathrm{Re}_L\theta_{11}=\mathrm{P}_r\mathrm{Re}_L v_{10}\frac{d\theta_{01}}{dy}-2\mathrm{P}_r\mathrm{Re}_L\frac{du_{oo}}{dy}\frac{du_{10}}{dy} \quad (21)$$

$$\frac{d^2u_{10}}{dy^2}-\mathrm{Re}_L v_{oo}\frac{du_{10}}{dy}-\mathrm{Re}_L M_1u_{10}=\mathrm{Re}_L v_{10}\frac{du_{oo}}{dy}-\mathrm{Re}_L M_1U_o-G_r\mathrm{Re}_L\theta_{10} \quad (22)$$

$$\frac{d^2u_{11}}{dy^2}-\mathrm{Re}_L v_{oo}\frac{du_{11}}{dy}-v_{01}\mathrm{Re}_L\frac{du_{10}}{dy}-M_1\mathrm{Re}_L u_{11}=-\mathrm{Re}_L\left(\frac{du_{01}}{dy}\right)+v_{11}\mathrm{Re}_L\left(\frac{du_{oo}}{dy}\right)-G_r\mathrm{Re}_L\theta_{11} \quad (23)$$

With the following sets of zeroth-order and first order boundary conditions as:

$$\left.\begin{array}{l}u_{oo}=u_p, u_{01}=0, v_{oo}=-1, u_{10}=0, u_{11}=0, v_{10}=-1, v_{11}=0,\\ v_{01}=0, \theta_{oo}=1, \theta_{01}=0, \theta_{10}=1, \theta_{11}=0 \text{ at } y=0, \text{ when } t>0\end{array}\right\} \quad (24.1)$$

$$\left.\begin{array}{l}u_{oo}\to U_o, u_{01}\to 0, u_{10}\to U_o, u_{11}\to 0, v_{10}=-1, v_{11}\to 0,\\ \theta_{10}\to 0, \theta_{11}\to 0, \theta_{oo}\to 0, \theta_{01}\to 0 \text{ for } y\to\infty,\ t>0\end{array}\right\} \quad (24.2)$$

Finally, the mean as well as the perturbed parts of the solutions for temperature and velocity fields are calculated as:

$$\theta_{oo}=\exp(-m_1y)$$

$$u_{oo}=U_o+b_2\exp(-m_1y)+b_3\exp(-m_2y)$$

$$\theta_{01}=a_4\exp(-m_1y)+a_1\exp(-2m_2y)+a_2\exp(-2m_1y)+a_3\exp(-\gamma_1y)$$

$$u_{01}=b_4\exp(-m_1y)+b_5\exp(-2m_2y)+b_6\exp(-2m_1y)+b_7\exp(-\gamma_1y)+b_8\exp(-m_2y)$$

$$\theta_{10R}=(1-a_6)\exp(-\alpha_1y)\cos\beta_1y+a_6\exp(-m_1y)-a_7\exp(-\alpha_1y)\sin\beta_1y$$

$$u_{10R}=U_o-(U_o+b_{12}+b_{15}+b_{19})\exp(-\alpha_2y)\cos\beta_2y-(b_{13}+b_{16}+b_{20})\exp(-\alpha_2y)\sin\beta_2y$$
$$+b_{12}\exp(-m_2y)+b_{15}\exp(-m_1y)+b_{19}\exp(-\alpha_1y)\cos\beta_1y+b_{20}\exp(-\alpha_1y)\sin\beta_1y$$

$$\theta_{11R} = a_9 \exp(-m_1 y) + a_{12}\exp(-2m_2 y) + a_{15}\exp(-2m_1 y) + a_{18}\exp(-\gamma_1 y) + a_{22}\exp(-(m_1+\alpha_1)y)\cos\beta_1 y + a_{23}\exp(-(m_1+\alpha_1)y)\sin\beta_1 y$$
$$+a_{26}\exp(-(m_1+\alpha_2)y)\cos\beta_2 y + a_{27}\exp(-(m_1+\alpha_2)y)\sin\beta_2 y + a_{30}\exp(-(m_2+\alpha_1)y)\cos\beta_1 y + a_{31}\exp(-(m_2+\alpha_1)y)\sin\beta_1 y$$
$$+a_{34}\exp(-(m_2+\alpha_2)y)\cos\beta_2 y + a_{35}\exp(-(m_2+\alpha_2)y)\sin\beta_2 y + a_{36}\exp(-\alpha_3 y)\cos\beta_3 y + a_{37}\exp(-\alpha_3 y)\sin\beta_3 y$$

$$u_{11R} = b_{36}\exp(-m_2 y) + b_{39}\exp(-m_1 y) + b_{42}\exp(-2m_2 y) + b_{45}\exp(-2m_1 y) + b_{48}\exp(-\gamma_1 y)$$
$$+b_{52}\exp(-(m_1+\alpha_1)y)\cos\beta_1 y + b_{53}\exp(-(m_1+\alpha_1)y)\sin\beta_1 y + b_{56}\exp(-(m_2+\alpha_1)y)\cos\beta_1 y$$
$$+b_{57}\exp(-(m_2+\alpha_1)y)\sin\beta_1 y + b_{60}\exp(-(m_2+\alpha_2)y)\cos\beta_2 y + b_{61}\exp(-(m_2+\alpha_2)y)\sin\beta_2 y$$
$$+b_{64}\exp(-(m_1+\alpha_2)y)\cos\beta_2 y + b_{56}\exp(-(m_2+\alpha_1)y)\cos\beta_1 y + b_{57}\exp(-(m_2+\alpha_1)y)\sin\beta_1 y$$
$$+b_{60}\exp(-(m_2+\alpha_2)y)\cos\beta_2 y + b_{61}\exp(-(m_2+\alpha_2)y)\sin\beta_2 y + b_{64}\exp(-(m_1+\alpha_2)y)\cos\beta_2 y$$
$$+b_{65}\exp(-(m_1+\alpha_2)y)\sin\beta_2 y + b_{68}\exp(-\alpha_3 y)\cos\beta_3 y + b_{69}\exp(-\alpha_3 y)\sin\beta_3 y$$
$$+b_{70}\exp(-\alpha_4 y)\cos\beta_4 y + b_{71}\exp(-\alpha_4 y)\sin\beta_4 y$$

Discussion on some quantities of engineering importance

Non-dimensional skin-friction $\tau_R = \dfrac{1}{\mathrm{Re}_L}(\tau_1 + (Ec\tau_2) + \varepsilon((\tau_3 + (Ec\tau_4))\cos\omega t - (\tau_5 + (Ec\tau_6))\sin\omega t))$

where, $\tau_1 = -(m_2 b_3 + m_1 b_2)$, $\tau_2 = -(m_1 b_4 + 2m_2 b_5 + 2m_1 b_6 + \gamma_1 b_7 + m_2 b_8)$

$$\tau_3 = \alpha_2(U_o + b_{12} + b_{15} + b_{19}) - \beta_2(b_{13} + b_{16} + b_{20}) - m_2 b_{12} - m_1 b_{15} - \alpha_1 b_{19} + \beta_1 b_{20}$$
$$\tau_4 = -\{m_2 b_{36} + m_1 b_{39} + 2m_2 b_{42} + 2m_1 b_{45} + \gamma_1 b_{48} + (m_1+\alpha_1)b_{52} - \beta_1 b_{53} + (m_2+\alpha_1)b_{56} - \beta_1 b_{57} + (m_2+\alpha_2)b_{60}$$
$$-\beta_2 b_{61} + (m_1+\alpha_2)b_{64} - \beta_2 b_{65} + \alpha_3 b_{68} - \beta_3 b_{69} + \alpha_4 b_{70} - \beta_4 b_{71}\}$$
$$\tau_5 = -\beta_2(U_o + b_{12} + b_{15} + b_{19}) - \alpha_2(b_{13} + b_{16} + b_{20}) + m_2 b_{13} + m_1 b_{16} + \beta_1 b_{19} + \alpha_1 b_{20}$$
$$\tau_6 = \{m_2 b_{37} + m_1 b_{40} + 2m_2 b_{43} + 2m_1 b_{46} + \gamma_1 b_{49} + \beta_1 b_{52} + (m_1+\alpha_1)b_{53} + \beta_1 b_{56} + (m_2+\alpha_1)b_{57} + \beta_2 b_{60} + (m_2+\alpha_2)b_{61}$$
$$+\beta_2 b_{64} + (m_1+\alpha_2)b_{65} + \beta_3 b_{68} + \alpha_3 b_{69} + \beta_4 b_{70} + \alpha_4 b_{71}\}$$

Non-dimensional heat transfer rate (Nusselt number)

$$Nu_R = -\frac{1}{P_r\,\mathrm{Re}_L}\left(\frac{\partial\theta_R}{\partial y}\right)_{y=0} = \frac{1}{P_r\,\mathrm{Re}_L}[Nu_1 + \varepsilon((Nu_2 + EcNu_3)\cos\omega t - (Nu_4 + EcNu_5)\sin\omega t)]$$

where, $Nu_1 = m_1 + Ec2m_2 a_1 + Ec2m_1 a_2 + Ec\gamma_1 a_3 + Ecm_1 a_4$, $Nu_2 = -\alpha_1 + \alpha_1 a_6 - m_1 a_6 - \beta_1 a_7$,

$$Nu_3 = -Ec(m_1 a_9 + 2m_2 a_{12} + 2m_1 a_{15} + \gamma_1 a_{18} + (m_1+\alpha_1)a_{22} - \beta_1 a_{23} + (m_1+\alpha_2)a_{26} - \beta_2 a_{27} + (m_2+\alpha_1)a_{30}$$
$$-\beta_1 a_{31} + (m_2+\alpha_2)a_{34} - \beta_2 a_{35} + \alpha_3 a_{36} - \beta_3 a_{37}),\quad Nu_4 = \beta_1 - \beta_1 a_6 - \alpha_1 a_7 + m_1 a_7$$
$$Nu_5 = Ec(m_1 a_{10} + 2m_2 a_{13} + 2m_1 a_{16} + \gamma_1 a_{19} + \beta_1 a_{22} + (m_1+\alpha_1)a_{23} + \beta_2 a_{26} + (m_1+\alpha_2)a_{27} + \beta_1 a_{30} + (m_2+\alpha_1)a_{31}$$
$$+\beta_2 a_{34} + (m_2+\alpha_2)a_{35} + \beta_3 a_{36} + \alpha_3 a_{37})$$

Results & Discussion

To get some physical insight into the problem, numerical simulation has been made to justify the influence of varies physical parameters that governed by the system due to inclusion of various physical situations, on the fluid variables, such as velocity, temperature and concentration as well as on some quantity of engineering interest like, skin-friction and Nusselt number.

In figures (1) to (5) the effect of pertinent parameters such as, Prandtl number (P_r), Permeability parameter (K), heat sink parameter (Q_S), thermal Grashof number (G_r) and thermal radiation parameter (N) on the temperature profile (θ_R, y) is depicted graphically. It is observed from these figures that the fluid temperature increases due to increase in parametric values of (K), (G_r), (N) while the increase of (P_r), (Q_S) found to be decreased the temperature within the thermal

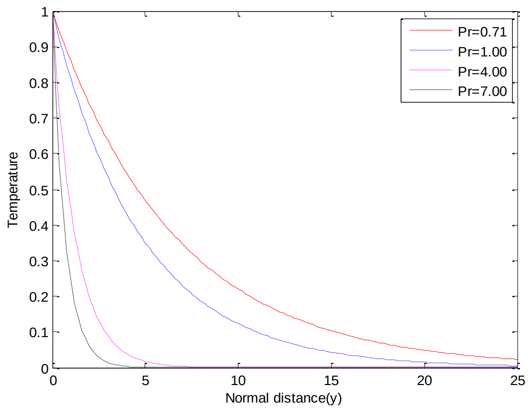

Fig. 1: Temperature against normal distances for arbitrary change in values of Prandtl number for Pr=7.0, Re_L=0.5, Q_S=0.01, N=2.5, G_r=5.0, U_0=0.5, u_p=0.5, ω=7.85714, ωt=1.57142857, K=0.5, ϵ=0.001, Ec=0.001.

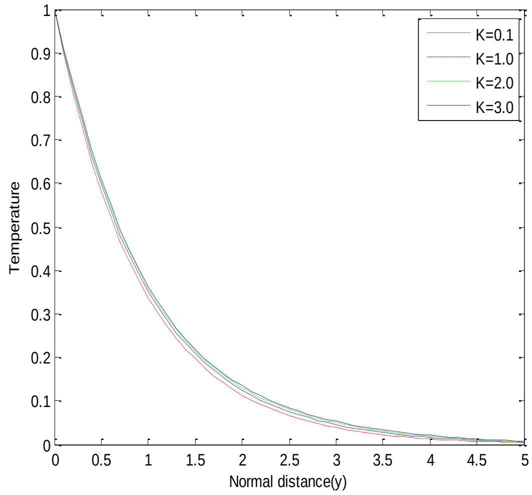

Fig. 2: Temperature against normal distances for arbitrary change in values of Permeability parameter for Pr=7.0, Re_L=0.5, Q_S=0.1, N=3.6, G_r=10.0, U_0=0.5, u_p=0.5, ω=7.85714, ωt=1.57142857, K=0.5, ϵ=0.001, Ec=0.01

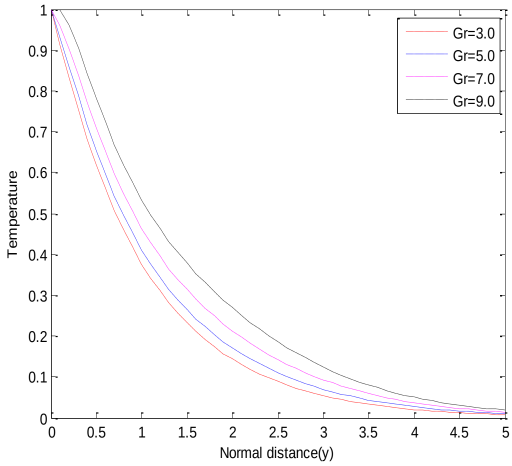

Fig. 4: Temperature against normal distances for arbitrary change in values of Thermal Grashof number for Pr=7.0, Re_L=0.05, Q_S=0.1, N=1.2, G_r=10.0, U_0=0.5, u_p=0.5, ω=7.85714, ωt=1.57142857, K=0.5, ϵ=0.001, Ec=0.01

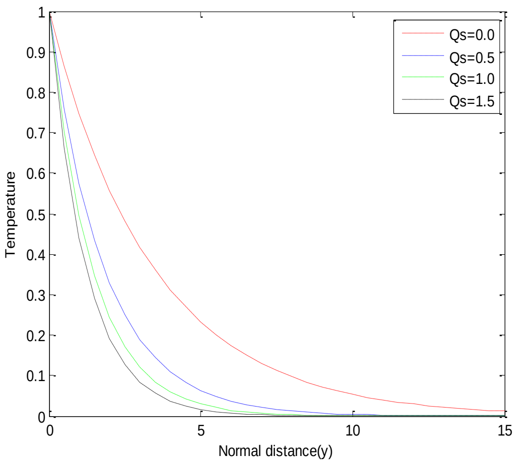

Fig. 3: Temperature against normal distances for arbitrary change in values of heat sink parameter for Pr=7.0, ReL=0.05, QS=0.1, N=1.2, Gr=5.0, U0=0.5, up=0.5, ω=7.85714, ωt=1.57142857, K=0.5, ϵ =0.001, Ec=0.001

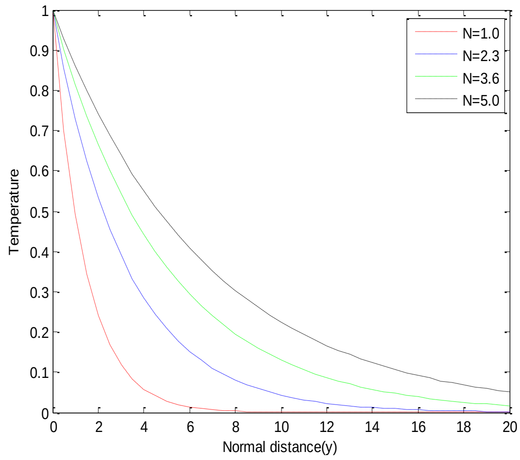

Fig.5: Temperature against normal distances for arbitrary change in values of Thermal radiation for Pr=7.0, Re_L=0.1, Q_S=0.01, N=1.2, G_r=1.0, U_0=0.5, u_p=0.5, ω=7.85714, ωt=1.57142857, K=0.5, ϵ=0.001, Ec=0.001

In Fig. 6, the influence of Eckert number (Ec) on the temperature profile (θ_R, y) is shown graphically. Due to increase in values of (Ec), the fluid experiences frictional heating in the intermediate layers; this thus, contributed in the thickening of thermal boundary layer and thus increases the temperature within the boundary layer significantly.

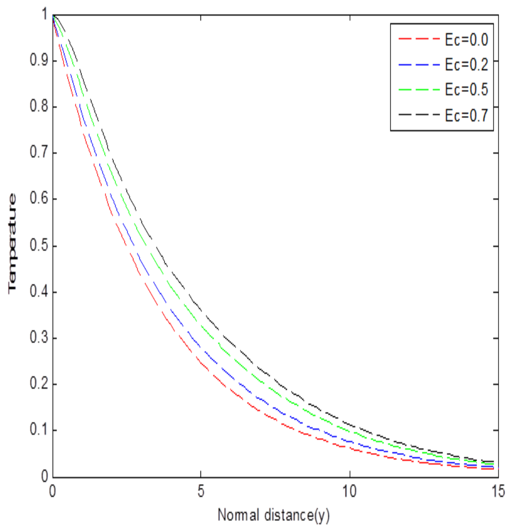

Fig. 6: Temperature against normal distances for arbitrary change in values of Eckert number for Pr=7.0, Re_L=0.1, Q_S=0.01, N=2.6, G_r=10.0, U_0=0.5, u_p=0.5, ω=7.85714, ωt=1.57142857, K=0.5, ϵ=0.001, Ec=0.001

The influence of Reynolds number (Re_L) on the temperature profile (θ_R, y) is demonstrated graphically in figure (7). The increase in values of (Re_L) increases the strength of the suction velocity on the plate which minimizes the growth of thermal boundary layer. This thus diminishes the temperature θ_R near to the plate surface.

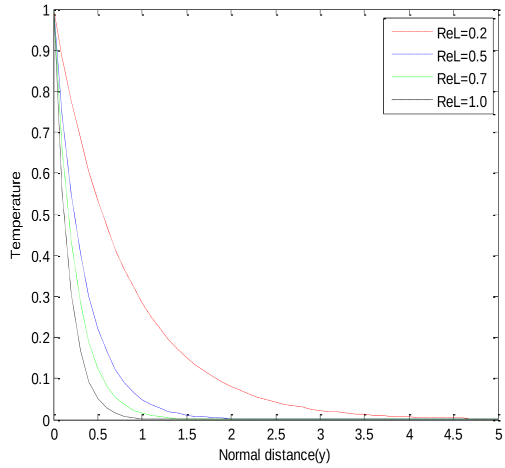

Fig. 7: Temperature against normal distances for arbitrary change in values of Local Reynolds number for Pr=7.0, Re_L=0.05, Q_S=0.1, N=1.2, G_r=5.0, U_0=0.5, u_p=0.5, ω=7.85714, ωt=1.57142857, K=0.5, ϵ=0.001, Ec=0.001

The parametric effect of the pertinent parameters such as Prandtl number (P_r) thermal Grashof number (G_r), heat absorption parameter (Q_S), permeability parameter (K) and thermal radiation parameter (N) on the velocity profiles (u_R, y) is demonstrated graphically interns of figures (8) to (12). The prime velocity is seen to increase due to increase in parametric values of (G_r) (K) and (N) while a reverse phenomenon as observed due to presence in values of (P_r) as well as (Q_S). The presence of (G_r) increases the thermal buoyancy force, which in turn accelerates the flow rate and thus increases the value of u_R. Due to rise in values of the permeability parameter, the resistance of the porous media increases, This results in increasing the velocity fluxes inside the momentum boundary layer, which accelerates the flow rate and increases the value of u_R. As the presence of thermal radiation parameter increases the temperature of the fluid particles near the plate surface, this results in effecting an increment in the flow rate by increasing the kinetic energy of the fluid particles as a results the value of u_R is found increasing.

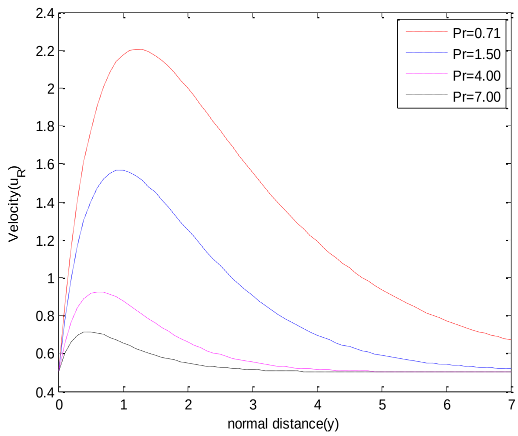

Fig. 8: Velocity against normal distances for arbitrary change in values of Prandtl number for Pr=7.0, Re_L=0.3, Q_S=0.5, N=2.6, G_r=1.0, U_0=0.5, u_p=0.5, ω=7.85714, ωt=1.57142857, K=0.5, ϵ=0.001, Ec=0.001

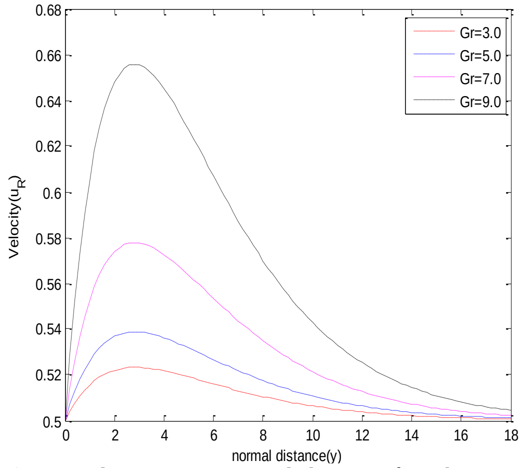

Fig. 9: Velocity against normal distances for arbitrary change in values of Thermal Grashof number for Pr=7.0, Re_L=0.05, Q_S=0.1, N=1.2, G_r=5.0, U_0=0.5, u_p=0.5, ω=7.85714, ωt=1.57142857, K=0.5, ϵ=0.001, Ec=0.001

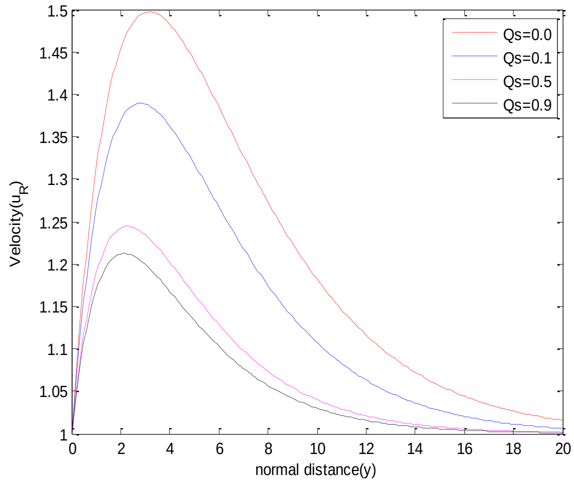

Fig. 10: Velocity against normal distances for arbitrary change in values of heat sink parameter for Pr=7.0, Re_L=0.05, Q_S=0.1, N=1.2, G_r=5.0, U_0=1.0, u_p=1.0, ω=7.85714, ωt=1.57142857, K=0.5, ϵ=0.001, Ec=0.001

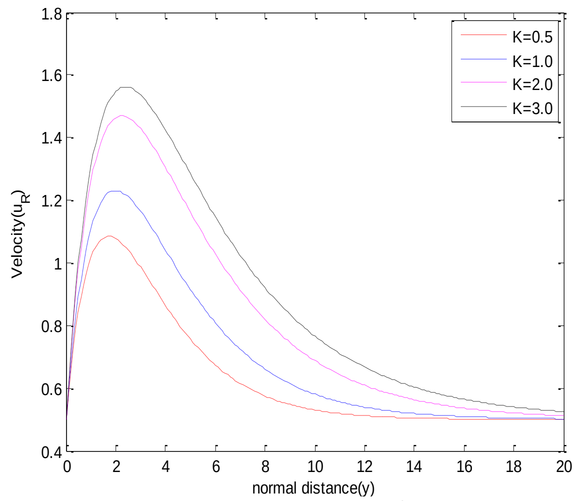

Fig. 11: Velocity against normal distances for arbitrary change in values of Permeability parameter for Pr=0.7, Re_L=0.1, Q_S=0.1, N=1.2, G_r=10.0, U_0=0.5, u_p=0.5, ω=7.85714, ωt=1.57142857, K=0.5, ϵ=0.001, Ec=0.01

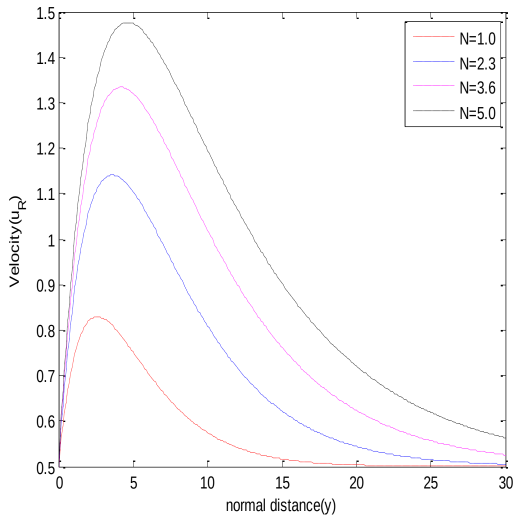

Fig. 12: Velocity against normal distances for arbitrary change in values of Thermal radiation for Pr=7.0, Re_L=0.05, Q_S=0.1, N=1.2, G_r=5.0, U_0=0.5, u_p=0.5, ω=7.85714, ωt=1.57142857, K=0.5, ϵ=0.001, Ec=0.001

The influence of physical parameters like Eckert number (Ec) and Reynolds number (Re_L) on the velocity profile (u_R, y) is shown graphically in figures (13) and (14) respectively. It is observed that, the velocity increases due to increase in values of (Ec) while a reverse phenomenon is seen due to presence of Re_L . As due to increase in values of *Ec*, frictional heating in the thermal boundary layer increases. This in turn increases the kinetic energy of the fluid particle in the intermediate layers as a result accelerates the flow rate and thus increases the velocity. Again due to raise in values of Re_L, the plate suction velocity increases. This in turn decreases the flow rate within the boundary layer and thus diminishes the velocity.

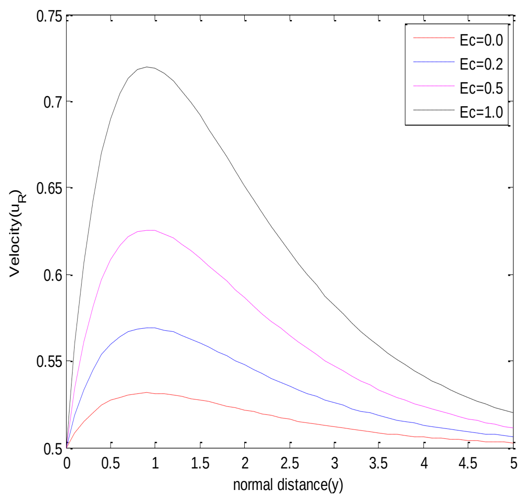

Fig. 13: Velocity against normal distances for arbitrary change in values of Eckert number for Pr=7.0, Re_L=0.2, Q_S=0.5, N=2.6, G_r=1.0, U_0=0.5, u_p=0.5, ω=7.85714, ωt=1.57142857, K=0.5, ϵ=0.001, Ec=0.001

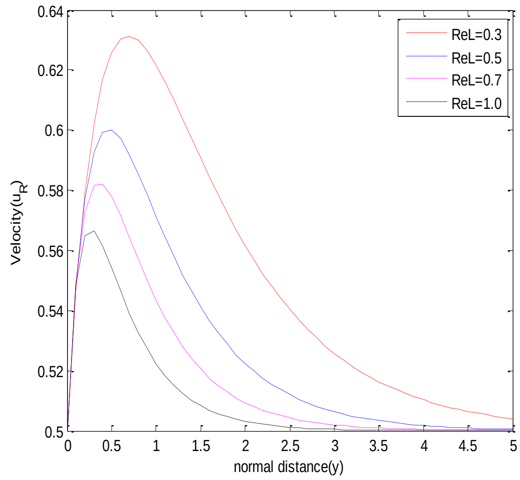

Fig. 14: Velocity against normal distances for arbitrary change in values of Local Reynolds number for Pr=7.0, Re_L=0.05, Q_S=0.5, N=1.2, G_r=5.0, U_0=0.5, u_p=0.5, ω=7.85714, ωt=1.57142857, K=0.5, ϵ=0.001, Ec=0.001

In figures (15) to (17), the influence of permeability parameter (*K*), heat absorption parameter (Q_s) and thermal radiation parameter (*N*) on the Nusselt number profiles (Nu_R, t) are demonstrated graphically. The Nusselt number is found decreasing due to increase in values of (*K*) and (*N*) respectively while, a reverse phenomenon is observed due to presence of (Q_s). Due to an increment in values of (*K*) and (*N*), the thickness of the thermal boundary layer increases, this restricts the heat transfer process to transmit plate region to free stream region results of which declines the heat transfer rate. This in turn decreases the value of Nu_R. Again due to thinning of thermal boundary layer affected by presence of heat absorption parameter, the heat fluxes transmit spontaneously towards the free stream region. This transmission of heat transfer effects in increasing the values of Nu_R.

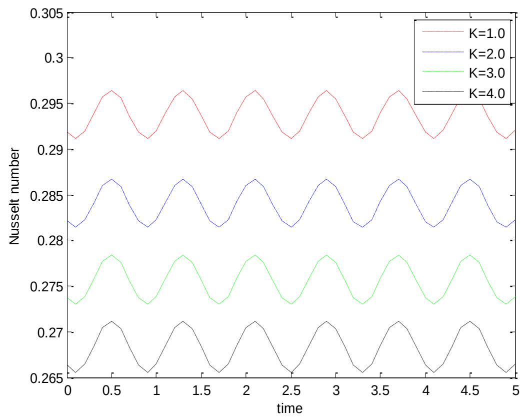

Fig. 15: Nusselt number against time for arbitrary change in values of Permeability parameter for Pr=7.0, Re_L=0.05, Q_S=0.01, N=3.6, G_r=10.0, U_0=1.0, u_p=1.0, ω=7.85714, ωt=1.57142857, K=0.5, ϵ=0.001, Ec=0.01

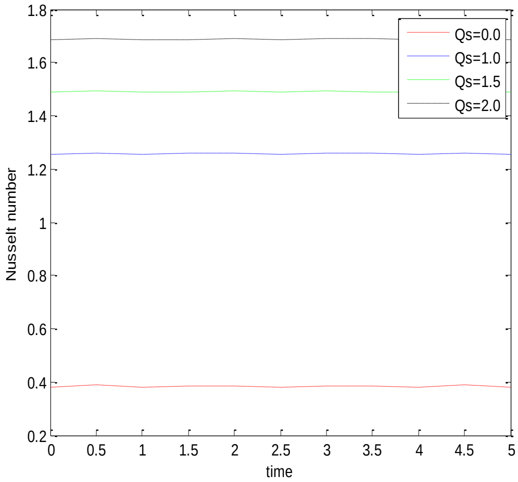

Fig. 16: Nusselt number against time for arbitrary change in values of heat sink parameter for Pr=7.0, Re_L=0.05, Q_S=0.1, N=2.6, G_r=8.0, U_0=0.5, u_p=0.5, ω=7.85714, ωt=1.57142857, K=0.5, ϵ=0.001, Ec=0.001

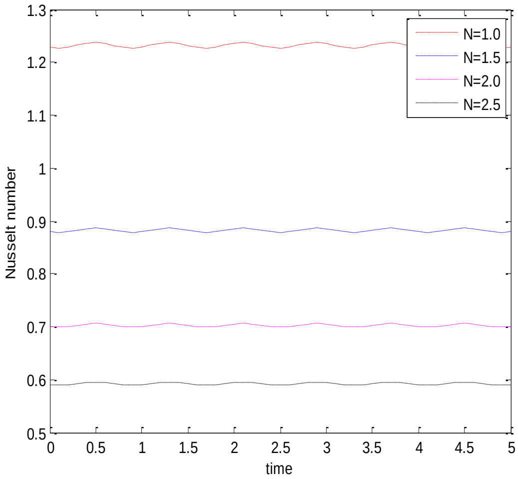

Fig. 17: Nusselt number against time for arbitrary change in values of Thermal radiation parameter for Pr=7.0, Re_L=0.05, Q_S=0.1, N=3.6, G_r=8.0, U_0=0.5, u_p=0.5, ω=7.85714, ωt=1.57142857, K=0.5, ϵ=0.001, Ec=0.001

Figures 18 to 20 show how the skin-friction τ_R against time t is being affected due to influence of parameters like heat absorption parameter (Q_S), permeability parameter (K) and thermal radiation parameter (N). It is clearly seen from these figures that, the skin-friction increases due to increase in parametric values of (K) and (N) but a reversed phenomenon are being observed due to presence of (Q_S). Due to an increase in value of (K), though the skin-friction shows an increasing trend, but it is clearly shows from figure 18 that, the skin-frictional values attain a steady state due to presence of (K). The increase of skin-friction values in presence of (K)and (N) is due to the fact that, the flow rate in both the cases accelerates, as a result the plate surface experiences a drag force opposite to the motion. This force is responsible for the enhancement of the skin-frictional values on the plate.

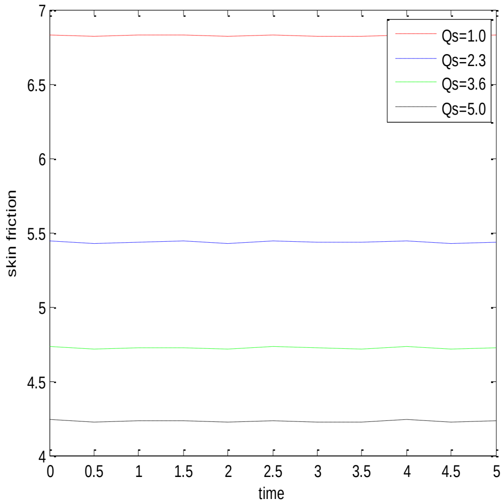

Fig. 18: Skin-friction against time for arbitrary change in values of heat sink parameter for Pr=7.0, Re_L=0.05, Q_S=0.01, N=2.6, G_r=5.0, U_0=0.5, u_p=0.5, ω=7.85714, ωt=1.57142857, K=0.5, ϵ=0.001, Ec=0.001

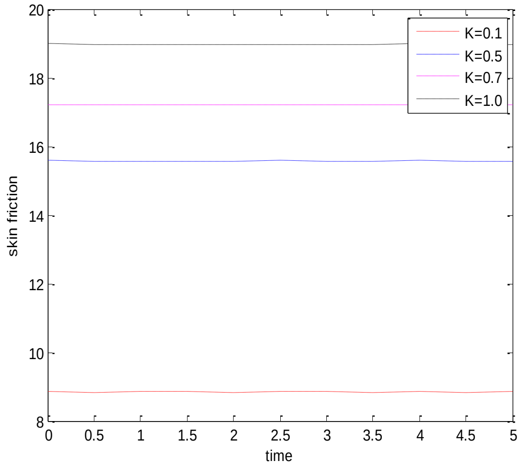

Fig. 19: Nusselt number against time for arbitrary change in values of Permeability parameter for Pr=7.0, Re_L=0.05, Q_S=0.1, N=2.3, G_r=8.0, U_0=0.5, u_p=0.5, ω=7.85714, ωt=1.57142857, K=0.5, ϵ=0.001, Ec=0.001

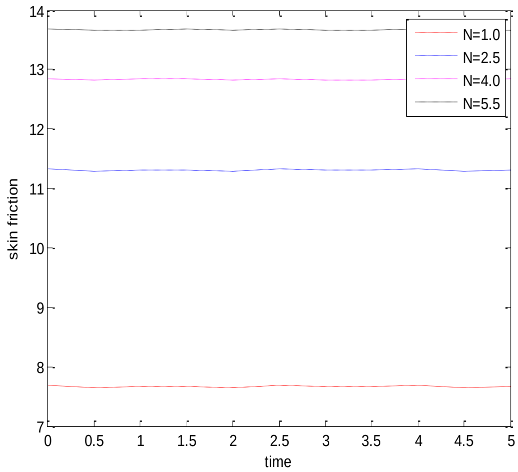

Fig. 20: Skin-friction against time for arbitrary change in values of Thermal radiation parameter for Pr=7.0, Re_L=0.05, Q_S=0.01, N=3.2, G_r=5.0, U_0=1.0, u_p=1.0, ω=7.85714, ωt=1.57142857, K=0.5, ϵ=0.001, Ec=0.001

In Table 1, a comparison of the present work with the work of Babu*et al*. (2011) is made to validate the effects of the Eckert number on the Local Nusselt number. It is observed that in both of these works, the Nusselt number decreases gradually with the increase of Eckert number against the fixed time t=1.0. The table shows a good agreement in the present work with that of Babu*et al*. (2011).Table 2 shows numerically the variations in values of skin-friction as well as the Nusselt number for arbitrary change in values of local Reynolds number against change in time. It is clearly shows that, the Skin-friction and the Nusselt number decreases steadily by increase in values of Reynolds number. It is also seen that, due to passage of time, both the skin-frictional values and that of the Nusselt number show a fluctuating trends.

Table 1: A comparison in the values of Nusselt number against Eckert number at t=1.0 for Pr=7.1, Re_L=1.0, Q_S=0.0, N=1.2, G_r=2.0, U_0=1.0, u_p=0.5, ω=0.1, K=0.5, ϵ=0.001, Ec=0.001

Ec	**Babu*et al.* (2011)**	**Present paper**
0.00	1.1376	0.8317
0.01	0.8652	0.8286
0.02	0.5429	0.8255
0.03	0.2546	0.8225

Table 2: Table displaying the variation in values of skin-friction and Nusselt number due to influence of local Reynolds number against time

Time	$Skin-friction(\tau_R)$				$Nusselt$ number(Nu_R)			
	$Re_L=0.5$	$Re_L=0.7$	$Re_L=1.0$	$Re_L=1.5$	$Re_L=0.5$	$Re_L=0.7$	$Re_L=1.0$	$Re_L=1.5$
0.0000	1.3293	1.0055	0.7543	0.5696	0.4574	0.4505	0.4453	0.4413
0.5000	1.3244	1.0012	0.7507	0.5666	0.4597	0.4525	0.4471	0.4429
1.0000	1.3279	1.0043	0.7534	0.5688	0.4580	0.4510	0.4458	0.4419
1.5000	1.3278	1.0042	0.7532	0.5686	0.4582	0.4511	0.4458	0.4417
2.0000	1.3244	1.0013	0.7508	0.5666	0.4596	0.4525	0.4471	0.4430
2.5000	1.3293	1.0055	0.7544	0.5696	0.4574	0.4505	0.4453	0.4413
3.0000	1.3258	1.0025	0.7518	0.5674	0.4591	0.4519	0.4466	0.4424
3.5000	1.3258	1.0026	0.7519	0.5675	0.4589	0.4519	0.4466	0.4426
4.0000	1.3293	1.0055	0.7543	0.5696	0.4574	0.4505	0.4453	0.4413
4.5000	1.3244	1.0012	0.7507	0.5666	0.4597	0.4525	0.4471	0.4429
5.0000	1.3279	1.0043	0.7534	0.5688	0.4580	0.4510	0.4459	0.4419

Table 3: Table displaying the variation in values of skin-friction and Nusselt number due to influence of Eckert number against time

Time	$Skin-friction(\tau_R)$				$Nusselt$ number(Nu_R)			
	$Ec=0.0$	$Ec=0.1$	$Ec=0.2$	$Ec=0.3$	$Ec=0.0$	$Ec=0.1$	$Ec=0.2$	$Ec=0.3$
0.0000	12.0870	11.5166	10.9461	10.3757	0.6290	0.6207	0.6124	0.6042
0.5000	12.0595	11.5000	10.9404	10.3809	0.6348	0.6265	0.6183	0.6101
1.0000	12.0788	11.5197	10.9605	10.4013	0.6294	0.6211	0.6128	0.6045
1.5000	12.0790	11.5084	10.9378	10.3672	0.6312	0.6230	0.6147	0.6064
2.0000	12.0594	11.5046	10.9498	10.3949	0.6340	0.6258	0.6175	0.6093
2.5000	12.0869	11.5213	10.9556	10.3899	0.6282	0.6199	0.6116	0.6034
3.000	12.0677	11.5016	10.9354	10.3693	0.6336	0.6254	0.6171	0.6089
3.5000	12.0673	11.5127	10.9581	10.4034	0.6318	0.6235	0.6153	0.6070
4.0000	12.0871	11.5167	10.9463	10.3759	0.6290	0.6207	0.6124	0.6041
4.5000	12.0596	11.4999	10.9403	10.3807	0.6348	0.6265	0.6183	0.6101
5.0000	12.0787	11.5196	10.9605	10.4015	0.6294	0.6211	0.6128	0.6046

In Table 3, the influence of Eckert number on the skin-friction and on the Nusselt number is depicted for arbitrary change in values of time. It is observed that, both the skin-friction and the Nusselt number are also showing a decreasing trend due to increase in Eckert number. As shown earlier, the skin-friction and the Nusselt number both exhibit a fluctuating trend due to increment of time variable in presence of Eckert number.

Conclusions

A mixed convective flow problem of Newtonian, incompressible, viscous fluid past a porous plate through Darcian porous media and under the influence of thermal buoyancy forces is studied in presence of viscous dissipation and thermal radiation. A multi - parameter perturbation scheme is developed to solve the governed equations and the influence of various physical parameters on the flow variables are numerically simulated and interpreted through graphs and tables. The significant outcomes of the investigation are as follows:

- The fluid temperature decreases due to increase in values of Reynolds number, Prandtl number and heat absorption parameters, while an increase in values of Eckert number, Grashof number, permeability parameter and thermal radiation parameter increases the fluid temperature.
- The increase in values of thermal Grashof number, permeability parameter, thermal radiation parameter and Eckert number increases the fluid velocity but a reversed phenomenon has observed on the velocity due to increase in parametric values of Prandtl number, heat source parameter and Reynolds number.

- The Nusselt number is found decreasing due to increase in values of permeability parameter, thermal radiation parameter, Reynolds number and Eckert number, while the Nusselt number is found fluctuating with time.
- The Skin-friction on the plate increases as thermal radiation and permeability parameters increase, but the skin-friction is found decreasing due to increase in heat sink parameter, Reynolds number and Eckert number. The skin-friction is also found in fluctuating mode against time.

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Appendix

$$m_1 = \frac{P_r \operatorname{Re}_L + \sqrt{(P_r \operatorname{Re}_L)^2 + 4P_r \operatorname{Re}_L Q_S N}}{2N},\ m_2 = \frac{\operatorname{Re}_L + \sqrt{\operatorname{Re}_L{}^2 + \frac{4\operatorname{Re}_L}{k}}}{2},\ b_1 = m_1{}^2 - m_1 \operatorname{Re}_L - \frac{\operatorname{Re}_L}{k}$$

$$b_2 = -\frac{G_r \operatorname{Re}_L}{b_1},\ b_3 = u_p - U_o - b_2,\ m_3 = \frac{P_r \operatorname{Re}_L + \sqrt{(P_r \operatorname{Re}_L)^2 + 4P_r \operatorname{Re}_L Q_S N}}{2N}$$

$$a_1 = \frac{-P_r \operatorname{Re}_L m_2{}^2 b_3{}^2}{4Nm_2{}^2 - 2m_2 P_r \operatorname{Re}_L - P_r \operatorname{Re}_L Q_s},\ a_2 = \frac{-P_r \operatorname{Re}_L m_1{}^2 b_2{}^2}{4Nm_1{}^2 - 2m_1 P_r \operatorname{Re}_L - P_r \operatorname{Re}_L Q_s},\ a_3 = \frac{-P_r \operatorname{Re}_L 2m_1 m_2 b_2 b_3}{N\gamma_1{}^2 - \gamma_1 P_r \operatorname{Re}_L - P_r \operatorname{Re}_L Q_s}$$

$$a_4 = -(a_1 + a_2 + a_3),\ \ \gamma_1 = m_1 + m_2 \in □,\ m_4 = \frac{\operatorname{Re}_L + \sqrt{\operatorname{Re}_L{}^2 + \frac{4\operatorname{Re}_L}{k}}}{2},\ b_4 = \frac{-G_r \operatorname{Re}_L a_4}{m_1{}^2 - m_1 \operatorname{Re}_L - \frac{\operatorname{Re}_L}{k}}$$

$$b_5 = \frac{-G_r \operatorname{Re}_L a_1}{4m_2{}^2 - 2m_2 \operatorname{Re}_L - \frac{\operatorname{Re}_L}{K}}$$

$$b_6 = -\frac{G_r \operatorname{Re}_L a_2}{4m_1{}^2 - 2m_1 \operatorname{Re}_L - \frac{\operatorname{Re}_L}{k}},\ b_7 = -\frac{G_r \operatorname{Re}_L a_3}{\gamma_1{}^2 - \gamma_1 \operatorname{Re}_L - \frac{\operatorname{Re}_L}{k}},\ b_8 = -(b_4 + b_5 + b_6 + b_7)$$

$$\alpha_1 = \frac{1}{2N}[P_r \operatorname{Re}_L + \frac{1}{\sqrt{2}}\sqrt{\left((P_r \operatorname{Re}_L)^2 + 4Q_S N P_r \operatorname{Re}_L\right) + \sqrt{\left((P_r \operatorname{Re}_L)^2 + 4Q_S N P_r \operatorname{Re}_L\right)^2 + 16\omega^2 N^2 (P_r \operatorname{Re}_L)^2}}\,]$$

$$\beta_1 = \frac{1}{2N}[\frac{1}{\sqrt{2}}\sqrt{\sqrt{\left((P_r \operatorname{Re}_L)^2 + 4Q_S N P_r \operatorname{Re}_L\right)^2 + 16\omega^2 N^2 (P_r \operatorname{Re}_L)^2} - (P_r \operatorname{Re}_L)^2 - 4Q_S N P_r \operatorname{Re}_L}\,],\ a_5 = Nm_1{}^2 - m_1 P_r \operatorname{Re}_L - P_r \operatorname{Re}_L Q_s$$

$$a_6 = \frac{m_1 a_5 P_r \operatorname{Re}_L}{a_5{}^2 + \omega^2 (P_r \operatorname{Re}_L)^2},\ a_7 = \frac{m_1 (P_r \operatorname{Re}_L)^2 \omega}{a_5{}^2 + \omega^2 (P_r \operatorname{Re}_L)^2},\ \alpha_2 = \frac{1}{2}[\operatorname{Re}_L + \frac{1}{\sqrt{2}}\sqrt{\left(\operatorname{Re}_L{}^2 + \frac{4\operatorname{Re}_L}{k}\right) + \sqrt{\left(\operatorname{Re}_L{}^2 + \frac{4\operatorname{Re}_L}{k}\right)^2 + 16\omega^2 \operatorname{Re}_L{}^2}}\,]$$

$$\beta_2 = \frac{1}{2}[\frac{1}{\sqrt{2}}\sqrt{\sqrt{(\operatorname{Re}_L{}^2 + \frac{4\operatorname{Re}_L}{K})^2 + 16\omega^2 \operatorname{Re}_L{}^2} - \operatorname{Re}_L{}^2 - \frac{4\operatorname{Re}_L}{K}}\,],\ b_9 = \operatorname{Re}_L m_1 b_2 - G_r \operatorname{Re}_L a_6$$

$$b_{10} = G_r \operatorname{Re}_L - G_r \operatorname{Re}_L a_6,\ b_{11} = m_2{}^2 - m_2 \operatorname{Re}_L - \frac{\operatorname{Re}_L}{k},\ b_{12} = \frac{\operatorname{Re}_L m_2 b_3 b_{11}}{b_{11}{}^2 + \omega^2 \operatorname{Re}_L{}^2},\ b_{13} = \frac{\operatorname{Re}_L{}^2 b_3 m_2 \omega}{b_{11}{}^2 + \omega^2 \operatorname{Re}_L{}^2}$$

$$b_{14} = m_1{}^2 - m_1 \operatorname{Re}_L - \frac{\operatorname{Re}_L}{k},\ b_{15} = \left(\frac{b_9 b_{14}}{b_{14}{}^2 + \omega^2 \operatorname{Re}_L{}^2} + \frac{G_r \operatorname{Re}_L{}^2 a_7 \omega}{b_{14}{}^2 + \omega^2 \operatorname{Re}_L{}^2}\right),\ b_{16} = \left(\frac{b_9 \omega \operatorname{Re}_L}{b_{14}{}^2 + \omega^2 \operatorname{Re}_L{}^2} - \frac{G_r \operatorname{Re}_L a_7 b_{14}}{b_{14}{}^2 + \omega^2 \operatorname{Re}_L{}^2}\right)$$

$$b_{17} = \alpha_1{}^2 - \beta_1{}^2 - \alpha_1 \operatorname{Re}_L - \frac{\operatorname{Re}_L}{k},\ b_{18} = 2\alpha_1\beta_1 - \operatorname{Re}_L \beta_1 - \omega \operatorname{Re}_L,\ b_{19} = \left(-\frac{b_{10} b_{17}}{b_{17}{}^2 + b_{18}{}^2} + \frac{G_r \operatorname{Re}_L a_7 b_{18}}{b_{17}{}^2 + b_{18}{}^2}\right)$$

$$b_{20} = \frac{b_{10} b_{18}}{b_{17}{}^2 + b_{18}{}^2} + \frac{G_r \operatorname{Re}_L a_7 b_{17}}{b_{17}{}^2 + b_{18}{}^2},\ \alpha_3 = \frac{1}{2N}[P_r \operatorname{Re}_L + \frac{1}{\sqrt{2}}\sqrt{\left((P_r \operatorname{Re}_L)^2 + 4Q_S N P_r \operatorname{Re}_L\right) + \sqrt{\left((P_r \operatorname{Re}_L)^2 + 4Q_S N P_r \operatorname{Re}_L\right)^2 + 16\omega^2 N^2 (P_r \operatorname{Re}_L)^2}}\,]$$

$$\beta_3 = \frac{1}{2N}[\sqrt{\sqrt{\left((P_r \operatorname{Re}_L)^2 + 4Q_S N P_r \operatorname{Re}_L\right)^2 + 16\omega^2 N^2 (P_r \operatorname{Re}_L)^2} - (P_r \operatorname{Re}_L)^2 - 4Q_S N P_r \operatorname{Re}_L}\,]$$

$$b_{21} = -(b_{12} + b_{15} + b_{19}),\ b_{22} = -(b_{13} + b_{16} + b_{20}),\ b_{23} = U_o \alpha_2 - \alpha_2 b_{21} + \beta_2 b_{22}$$

$$b_{24} = U_o \beta_2 - \alpha_2 b_{22} - \beta_2 b_{21},\ b_{25} = \alpha_1 b_{19} - \beta_1 b_{20},\ b_{26} = \alpha_1 b_{20} + \beta_1 b_{19}$$

$$b_{27} = m_1 m_2 b_3 b_{15} + m_1 m_2 b_2 b_{12},\ b_{28} = m_1 m_2 b_3 b_{16} + m_1 m_2 b_2 b_{13}$$

$$\gamma_2 = m_1 + \alpha_1,\ \gamma_3 = m_1 + \alpha_2,\ \gamma_4 = m_2 + \alpha_1,\ \gamma_5 = m_2 + \alpha_2$$

$$\gamma_6 = \gamma_2 + i\beta_1,\ \gamma_7 = \gamma_4 + i\beta_1,\ \gamma_8 = \gamma_5 + i\beta_2,\ \gamma_9 = \gamma_3 + i\beta_2$$

$$b_{29} = 2m_2 a_1 P_r \operatorname{Re}_L - 2P_r \operatorname{Re}_L m_2{}^2 b_3 b_{12},\ b_{30} = -2P_r \operatorname{Re}_L m_2{}^2 b_3 b_{13},\ b_{31} = 2m_1 a_2 P_r \operatorname{Re}_L - 2P_r \operatorname{Re}_L m_1{}^2 b_2 b_{15}$$

$$b_{32} = -2P_r \operatorname{Re}_L m_1{}^2 b_2 b_{16},\ b_{33} = \gamma_1 a_3 P_r \operatorname{Re}_L - 2P_r \operatorname{Re}_L b_{27},\ b_{34} = -2P_r \operatorname{Re}_L b_{28},\ a_8 = Nm_1{}^2 - m_1 P_r \operatorname{Re}_L - Q_S P_r \operatorname{Re}_L$$

$$a_9 = \frac{P_r \operatorname{Re}_L m_1 a_4 a_8}{a_8^{\ 2} + \omega^2 (P_r \operatorname{Re}_L)^2},\ a_{10} = \frac{(P_r \operatorname{Re}_L)^2 m_1 a_4 \omega}{a_8^{\ 2} + \omega^2 \left(P_r \operatorname{Re}_L\right)^2}, a_{11} = \left(N4m_2^{\ 2} - 2m_2 P_r \operatorname{Re}_L - Q_S P_r \operatorname{Re}_L\right)$$

$$a_{12} = \frac{a_{11} b_{29}}{a_{11}^{\ 2} + \omega^2 \left(P_r \operatorname{Re}_L\right)^2} - \frac{\left(P_r \operatorname{Re}_L\right) b_{30} \omega}{a_{11}^{\ 2} + \omega^2 \left(P_r \operatorname{Re}_L\right)^2}, a_{13} = \left(\frac{P_r \operatorname{Re}_L \omega b_{29}}{a_{11}^{\ 2} + \omega^2 \left(P_r \operatorname{Re}_L\right)^2} + \frac{b_{30} a_{11}}{a_{11}^{\ 2} + \omega^2 \left(P_r \operatorname{Re}_L\right)^2}\right)$$

$$a_{14} = N4m_1^{\ 2} - 2m_1 P_r \operatorname{Re}_L - Q_S P_r \operatorname{Re}_L, a_{15} = \left(\frac{a_{14} b_{31}}{a_{14}^{\ 2} + \omega^2 \left(P_r \operatorname{Re}_L\right)^2} - \frac{\left(P_r \operatorname{Re}_L\right) \omega b_{32}}{a_{14}^{\ 2} + \omega^2 \left(P_r \operatorname{Re}_L\right)^2}\right)$$

$$a_{16} = \left(\frac{P_r \operatorname{Re}_L b_{31} \omega}{a_{14}^{\ 2} + \omega^2 \left(P_r \operatorname{Re}_L\right)^2} + \frac{b_{32} a_{14}}{a_{14}^{\ 2} + \omega^2 \left(P_r \operatorname{Re}_L\right)^2}\right), a_{17} = N\gamma_1^{\ 2} - \gamma_1 P_r \operatorname{Re}_L - P_r \operatorname{Re}_L Q_S$$

$$a_{18} = \left(\frac{a_{17} b_{33}}{a_{17}^{\ 2} + \omega^2 \left(P_r \operatorname{Re}_L\right)^2} - \frac{\omega \left(P_r \operatorname{Re}_L\right) b_{34}}{a_{17}^{\ 2} + \omega^2 \left(P_r \operatorname{Re}_L\right)^2}, a_{19} = \left[\frac{b33 \omega P_r \operatorname{Re}_L}{a_{17}^{\ 2} + \omega^2 \left(P_r \operatorname{Re}_L\right)^2} + \frac{b_{34} a_{17}}{a_{17}^{\ 2} + \omega^2 \left(P_r \operatorname{Re}_L\right)^2}\right]\right.$$

$$a_{20} = N\{(m_1 + \alpha_1)^2 - \beta_1^{\ 2}\} - P_r \operatorname{Re}_L (m_1 + \alpha_1) - P_r \operatorname{Re}_L,\ a_{21} = 2N\beta_1 (m_1 + \alpha_1) - \beta_1 P_r \operatorname{Re}_L - \omega P_r \operatorname{Re}_L$$

$$a_{22} = -\frac{2P_r \operatorname{Re}_L m_1 b_2 b_{25} a_{20}}{a_{20}^{\ 2} + a_{21}^{\ 2}} - \frac{2P_r \operatorname{Re}_L m_1 b_2 b_{26} a_{21}}{a_{20}^{\ 2} + a_{21}^{\ 2}},\ a_{23} = \frac{2P_r \operatorname{Re}_L m_1 b_2 b_{25} a_{21}}{a_{20}^{\ 2} + a_{21}^{\ 2}} - \frac{2P_r \operatorname{Re}_L m_1 b_2 b_{26} a_{20}}{a_{20}^{\ 2} + a_{21}^{\ 2}}$$

$$a_{24} = N\{(m_1 + \alpha_2)^2 - \beta_2^{\ 2}\} - P_r \operatorname{Re}_L (m_1 + \alpha_2) - P_r \operatorname{Re}_L Q_s,\ a_{25} = 2N\beta_2 (m_1 + \alpha_2) - \beta_2 P_r \operatorname{Re}_L - P_r \operatorname{Re}_L \omega$$

$$a_{26} = \frac{2P \operatorname{Re}_L m_1 b_2 b_{23} a_{24}}{a_{24}^{\ 2} + a_{25}^{\ 2}} + \frac{2P_r \operatorname{Re}_L m_1 b_2 b_{24} a_{25}}{a_{24}^{\ 2} + a_{25}^{\ 2}},\ a_{27} = -\frac{2P_r \operatorname{Re}_L m_1 b_2 b_{23} a_{25}}{a_{24}^{\ 2} + a_{25}^{\ 2}} + \frac{2P_r \operatorname{Re}_L m_1 b_2 b_{24} a_{24}}{a_{24}^{\ 2} + a_{25}^{\ 2}}$$

$$a_{28} = N\{(m_2 + \alpha_1)^2 - \beta_1^{\ 2}\} - P_r \operatorname{Re}_L (m_2 + \alpha_1) - P_r \operatorname{Re}_L Q_s,\ a_{29} = 2N\beta_1 (m_2 + \alpha_1) - \beta_1 P_r \operatorname{Re}_L - P_r \operatorname{Re}_L \omega$$

$$a_{30} = -\frac{2P_r \operatorname{Re}_L m_2 b_3 b_{25} a_{28}}{a_{28}^{\ 2} + a_{29}^{\ 2}} - \frac{2P_r \operatorname{Re}_L m_2 b_3 b_{26} a_{29}}{a_{28}^{\ 2} + a_{29}^{\ 2}},\ a_{31} = \frac{2P_r \operatorname{Re}_L m_2 b_3 b_{25} a_{29}}{a_{28}^{\ 2} + a_{29}^{\ 2}} - \frac{2P_r \operatorname{Re}_L m_2 b_3 b_{26} a_{28}}{a_{28}^{\ 2} + a_{29}^{\ 2}}$$

$$a_{32} = N\{(m_2 + \alpha_2)^2 - \beta_2^{\ 2}\} - (m_2 + \alpha_2) P_r \operatorname{Re}_L - P_r \operatorname{Re}_L Q_s,\ a_{33} = N\{(m_2 + \alpha_2)^2 - \beta_2^{\ 2}\} - P_r \operatorname{Re}_L (m_2 + \alpha_2) - P_r \operatorname{Re}_L Q_s$$

$$a_{34} = \frac{2P_r \operatorname{Re}_L m_2 b_3 b_{23} a_{32}}{a_{32}^{\ 2} + a_{33}^{\ 2}} + \frac{2P_r \operatorname{Re}_L m_2 b_3 b_{24} a_{33}}{a_{32}^{\ 2} + a_{33}^{\ 2}},\ a_{35} = -\frac{2P_r \operatorname{Re}_L m_2 b_3 b_{23} a_{33}}{a_{32}^{\ 2} + a_{33}^{\ 2}} + \frac{2P_r \operatorname{Re}_L m_2 b_3 b_{24} a_{32}}{a_{32}^{\ 2} + a_{33}^{\ 2}}$$

$$a_{36} = -(a_9 + a_{12} + a_{15} + a_{18} + a_{22} + a_{26} + a_{30} + a_{34}),\ a_{37} = -(a_{10} + a_{13} + a_{16} + a_{19} + a_{23} + a_{27} + a_{31} + a_{35})$$

$$\alpha_4 = \frac{1}{2}\left[\operatorname{Re}_L + \frac{1}{\sqrt{2}} \sqrt{\left(\operatorname{Re}_L^{\ 2} + \frac{4\operatorname{Re}_L}{k}\right) + \sqrt{\left(\operatorname{Re}_L^{\ 2} + \frac{4\operatorname{Re}_L}{k}\right)^2 + 16\omega^2 \operatorname{Re}_L^{\ 2}}}\ \right]$$

$$\beta_4 = \frac{1}{2}\left[\frac{1}{\sqrt{2}} \sqrt{\sqrt{\left(\operatorname{Re}_L^{\ 2} + \frac{4\operatorname{Re}_L}{k}\right)^2 + 16\omega^2 \operatorname{Re}_L^{\ 2}} - \operatorname{Re}_L^{\ 2} - \frac{4\operatorname{Re}_L}{k}}\ \right],\ a_{38} = \operatorname{Re}_L 2m_2 b_5 - Gr \operatorname{Re}_L a_{12}$$

$$a_{39} = -Gr \operatorname{Re}_L a_{13},\ a_{40} = \operatorname{Re}_L 2m_1 b_6 - Gr \operatorname{Re}_L a_{15},\ a_{41} = -Gr \operatorname{Re}_L a_{16},\ a_{42} = \operatorname{Re}_L \gamma_1 b_7 - Gr \operatorname{Re}_L a_{18}$$

$$a_{43} = -Gr \operatorname{Re}_L a_{19},\ b_{35} = m_2^{\ 2} - m_2 \operatorname{Re}_L - \frac{\operatorname{Re}_L}{k},$$

$$b_{36} = \frac{\operatorname{Re}_L m_2 b_8 b_{35}}{b_{35}^{\ 2} + \omega^2 \operatorname{Re}_L^{\ 2}},\ b_{37} = \frac{\operatorname{Re}_L^{\ 2} m_2 b_8 \omega}{b_{35}^{\ 2} + \omega^2 \operatorname{Re}_L^{\ 2}}$$

$$b_{38} = m_1^{\ 2} - m_1 \operatorname{Re}_L - \frac{\operatorname{Re}_L}{k},$$

$$b_{39} = \frac{\operatorname{Re}_L m_1 b_4 b_{38}}{b_{38}^{\ 2} + \omega^2 \operatorname{Re}_L^{\ 2}},\ b_{40} = \frac{\operatorname{Re}_L^{\ 2} m_1 b_4 \omega}{b_{38}^{\ 2} + \omega^2 \operatorname{Re}_L^{\ 2}},$$

$$b_{41} = 4m_2^{\ 2} - 2m_2 \operatorname{Re}_L - \frac{\operatorname{Re}_L}{k}$$

$$b_{42} = \frac{a_{38} b_{41}}{b_{41}^{\ 1} + \omega^2 \operatorname{Re}_L^{\ 2}} - \frac{a_{39} \omega \operatorname{Re}_L}{b_{41}^{\ 1} + \omega^2 \operatorname{Re}_L^{\ 2}},\ b_{43} = \frac{a_{38} \omega \operatorname{Re}_L}{b_{41}^{\ 2} + \omega^2 \operatorname{Re}_L^{\ 2}} + \frac{a_{39} b_{41}}{b_{41}^{\ 2} + \omega^2 \operatorname{Re}_L^{\ 2}}$$

$$b_{44} = 4m_1^{\ 2} - 2m_1 \operatorname{Re}_L - \frac{\operatorname{Re}_L}{K},$$

$b_{45}=\dfrac{a_{40}b_{44}}{b_{44}^{2}+\omega^{2}\operatorname{Re}_L^{2}}-\dfrac{a_{41}\omega\operatorname{Re}_L}{b_{44}^{2}+\omega^{2}\operatorname{Re}_L^{2}}$, $b_{46}=\dfrac{a_{40}\omega\operatorname{Re}_L}{b_{44}^{2}+\omega^{2}\operatorname{Re}L^{2}}+\dfrac{a_{41}b_{44}}{b_{44}^{2}+\omega^{2}\operatorname{Re}L^{2}}$

$b_{47}=\gamma_1^{2}-\gamma_1\operatorname{Re}_L-\dfrac{\operatorname{Re}_L}{K}$,

$b_{48}=\dfrac{a_{42}b_{47}}{b_{47}^{2}+\omega^{2}\operatorname{Re}_L^{2}}-\dfrac{a_{43}\omega\operatorname{Re}_L}{b_{47}^{2}+\omega^{2}\operatorname{Re}_L^{2}}$, $b_{49}=\dfrac{a_{42}\omega\operatorname{Re}_L}{b_{47}^{2}+\omega^{2}\operatorname{Re}_L^{2}}+\dfrac{a_{43}b_{47}}{b_{47}^{2}+\omega^{2}\operatorname{Re}_L^{2}}$

$b_{50}=\{(m_1+\alpha_1)^{2}-\beta_1^{2}\}-\operatorname{Re}_L(m_1+\alpha_1)-\dfrac{\operatorname{Re}_L}{k}$, $b_{51}=2\beta_1(m_1+\alpha_1)-\beta_1\operatorname{Re}_L-\omega\operatorname{Re}_L$

$b_{52}=\dfrac{a_{22}b_{50}}{b_{50}^{2}+b_{51}^{2}}+\dfrac{a_{23}b_{51}}{b_{50}^{2}+b_{51}^{2}}$, $b_{53}=-\dfrac{a_{22}b_{51}}{b_{50}^{2}+b_{51}^{2}}+\dfrac{a_{23}b_{50}}{b_{50}^{2}+b_{51}^{2}}$, $b_{54}=\{(m_1+\alpha_2)^{2}-\beta_2^{2}\}-\operatorname{Re}_L(m_1+\alpha_2)-\dfrac{\operatorname{Re}_L}{k}$

$b_{55}=2\beta_2(m_1+\alpha_2)-\beta_2\operatorname{Re}_L-\omega\operatorname{Re}_L$, $b_{56}=\dfrac{a_{26}b_{54}}{b_{54}^{2}+b_{55}^{2}}+\dfrac{a_{27}b_{55}}{b_{54}^{2}+b_{55}^{2}}$, $b_{57}=-\dfrac{a_{26}b_{55}}{b_{54}^{2}+b_{55}^{2}}+\dfrac{a_{27}b_{54}}{b_{54}^{2}+b_{55}^{2}}$

$b_{58}=\{(m_2+\alpha_1)^{2}-\beta_1^{2}\}-\operatorname{Re}_L(m_2+\alpha_1)-\dfrac{\operatorname{Re}_L}{k}$, $b_{59}=2\beta_1(m_2+\alpha_1)-\beta_1\operatorname{Re}_L-\omega\operatorname{Re}_L$

$b_{60}=\dfrac{a_{30}b_{58}}{b_{58}^{2}+b_{59}^{2}}+\dfrac{a_{31}b_{59}}{b_{58}^{2}+b_{59}^{2}}$, $b_{61}=-\dfrac{a_{30}b_{59}}{b_{58}^{2}+b_{59}^{2}}+\dfrac{a_{31}b_{58}}{b_{58}^{2}+b_{59}^{2}}$, $b_{62}=\{(m_2+\alpha_2)^{2}-\beta_2^{2}\}-\operatorname{Re}_L(m_2+\alpha_2)-\dfrac{\operatorname{Re}_L}{k}$

$b_{63}=2\beta_2(m_2+\alpha_2)-\beta_2\operatorname{Re}_L-\omega\operatorname{Re}_L$,

$b_{64}=\dfrac{a_{34}b_{62}}{b_{62}^{2}+b_{63}^{2}}+\dfrac{a_{35}b_{63}}{b_{62}^{2}+b_{63}^{2}}$, $b_{65}=-\dfrac{a_{34}b_{63}}{b_{62}^{2}+b_{63}^{2}}+\dfrac{a_{35}b_{62}}{b_{62}^{2}+b_{63}^{2}}$

$b_{66}=\alpha_3^{2}-\beta_3^{2}-\alpha_3\operatorname{Re}_L-\dfrac{\operatorname{Re}_L}{k}$, $b_{67}=2\alpha_3\beta_3-\beta_3\operatorname{Re}_L-\omega\operatorname{Re}_L$, $b_{68}=\dfrac{a_{36}b_{66}}{b_{66}^{2}+b_{67}^{2}}+\dfrac{a_{37}b_{67}}{b_{66}^{2}+b_{67}^{2}}$

$b_{69}=-\dfrac{a_{36}b_{67}}{b_{66}^{2}+b_{67}^{2}}+\dfrac{a_{37}b_{66}}{b_{66}^{2}+b_{67}^{2}}$, $b_{70}=-(b_{36}+b_{39}+b_{42}+b_{45}+b_{48}+b_{52}+b_{56}+b_{60}+b_{64}+b_{68})$

$b_{71}=-(b_{37}+b_{40}+b_{43}+b_{46}+b_{49}+b_{53}+b_{57}+b_{61}+b_{65}+b_{69})$.